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Automorphisms of the AT4(6, 6, 3)-graph and its Strongly-regular Graphs

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Koolen and Jurisich defined class of AT4-graphs (tight antipodal graph of diameter 4). Among these graphs available graph with intersection array $\{288, 245, 48, 1; 1, 24, 245, 288\}$ *on* $v = 1 + 288 + 2940 + 576 + 2 = 3807$ *vertices. Antipodal quotient of this graph is strongly regular graph with parameters* $(1269, 288, 42, 72)$. *Both these graphs are locally pseudo* $GQ(7, 5)$*-graphs. In this paper we find possible automorphisms of these graphs. In particular, group of automorphisms of distance-regular graph with intersection array* $\{288, 245, 48, 1; 1, 24, 245, 288\}$ *acts intransitive on the set of its antipodal classes.*

Keywords: distance-regular graph, strongly-regular graph, automorphism of the graph.
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1. Introduction and preliminaries

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at a distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex* a. Let $\Gamma(a) = \Gamma_1(a)$, $a^\perp = \{a\} \cup \Gamma(a)$. If graph Γ is fixed, then instead of $\Gamma(a)$ we write $[a]$. For the set of vertices X of graph Γ through $X^\perp$ denote $\Cap_{x\in X} x^\perp$.

Let $\mathcal{F}$ is some class of graphs. Graph Γ is called *locally* $\mathcal{F}$*-graph*, if $[a]$ lies in $\mathcal{F}$ for each vertex a of graph Γ. If the class $\mathcal{F}$ is composed of graphs, isomorphic to a certain graph Δ, then graph Γ is called *locally* Δ*-graph.*

Number of verices in neighbourhood of vertex is called *degree of a vertex.* Graph Γ is called *regular* with degree k, if the degree of any vertex a in Γ is k. Graph Γ is called *edge-regular* with parameters (v, k, λ), if it has v vertices, and it is regular with valency k, and each edge

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lies in λ triangles. Graph Γ is *amply regular graph* with parameters (v,k,λ,μ), if it is edge-regular with appropriate parameters and $[a]\cap[b]$ contains μ vertices for any two vertices a,b at distance 2 in Γ. Amply regular graph is called *strongly regular graph*, if it has diameter 2. Let $K_{m_1,\dots,m_n}$ be a complete multipartite graph $\{M_1,\dots,M_n\}$ with cocliques M_i of order m_i. If $m_1=\dots=m_n=m$, then this graph is denoted by $K_{n\times m}$.

Koolen and Jurisich [1] defined class of $AT4$-graphs (tight antipodal graph of diameter 4). Among these graphs available locally pseudo $GQ(7,5)$-graph with intersection array $\{288,245,48,1;1,24,245,288\}$, corresponding to $AT4(6,6,3)$-graph.

Let Γ be a distance-regular graph with intersection array $\{288,245,48,1;1,24,245,288\}$. Then antipodal quotient $\bar{\Gamma}$ is strongly regular graph with parameters $(1269,288,42,72)$. Both these graphs are locally pseudo $GQ(7,5)$-graphs. In this paper we have found possible automorphisms of these graphs.

Theorem 1.1. *Let Γ be a strongly regular graph with parameters $(288,42,6,6)$, $G=\mathrm{Aut}(\Gamma)$, g be an elment in G with prime order p and $\Omega=\mathrm{Fix}(g)$. Then $\pi(G)\subseteq\{2,3,5,7\}$ and one of the following statements holds:*

(1) *Ω is empty graph and either $p=2$, $\alpha_1(g)=24s$, or $p=3$, $\alpha_1(g)=36t$;*

(2) *Ω is n-clique and either $p=7$, $n=1$, $\alpha_1(g)=84s+42$ or $n=8$, $\alpha_1(g)=84s$, or $p=5$, $n=3$, $\alpha_1(g)=60s+30$ or $n=8$, $\alpha_1(g)=60s$;*

(3) *Ω is l-coclique and either $p=2$, l is even, $6\leqslant l\leqslant 36$ and $\alpha_1(g)=24t-6l$, or $p=3$, l is divided by 3, $3\leqslant l\leqslant 36$ and $\alpha_1(g)=36t-6l$;*

(4) *Ω contains geodesic 2-path, Ω does not have vertices of degree 42 and either*
(i) $p\leqslant 5$ and if $p=5$, then $|\Omega|=5t+3$, $t=2,3,\dots,9$, or
(ii) $p=3$, $|\Omega|=3s$, $s=2,3,\dots,16$, and in the case $s=2$ Ω is complete bipartite graph $K_{3,3}$, or
(iii) $p=2$, $|\Omega|=2l$, $l=2,3,\dots,24$.

Theorem 1.2. *Let Γ be a strongly regular graph with parameters $(1269,288,42,72)$, in which neighbourhoods of vertices are strongly regular graphs with parameters $(288,42,6,6)$, $G=\mathrm{Aut}(\Gamma)$, g be an elment in G with prime order p and $\Omega=\mathrm{Fix}(g)$. Then $\pi(G)\subseteq\{2,3,5,7,47\}$ and one of the following statements holds:*

(1) *Ω is empty graph and either $p=47$ and $\alpha_1(g)=47\cdot 6$, or $p=3$ and $\alpha_1(g)=126l+72$;*

(2) *if $[a]\subset\Omega$, then $\alpha_1(g)=0$, $a^\perp=\Omega$ and $p=2$;*

(3) *Ω is n-clique and either $p=2$, $n=1$, $\alpha_1(g)=84t+36$, or $p=5$, $n=4$, $\alpha_1(g)=210s+180$ or $n=9$, $\alpha_1(g)=210s$, or $p=7$, $n=2$, $\alpha_1(g)=294t+42$ or $n=9$, $\alpha_1(g)=294t+84$;*

(4) *Ω is l-coclique and either $p=2$, l is odd, $\alpha_1(g)=84m+6l+282$, or $p=3$, l divided by 3, $\alpha_1(g)=126m+6l+324$;*

(5) *Ω contains geodesic 2-path and either*
(i) $p=5$, $|\Omega|=5l+4$, $l\leqslant 64$, for $e\in\Omega$ we have $|\Omega(e)|=5t+3$, $t=2,3,\dots,9$, $\alpha_1(g)=210m+30l+180$, or
(ii) $p=3$, $|\Omega|=3n$, $n\leqslant 108$, for $e\in\Omega$ we have $|\Omega(e)|=3t$, $t=1,2,\dots,8$, $\alpha_1(g)=18n+126l+324$, or
(iii) $p=2$, $|\Omega|=2l+1$, $l\leqslant 161$, for $e\in\Omega$ we have $|\Omega(e)|=2n$, $n\leqslant 12$, $\alpha_1(g)=12l+84s+288$.

Corollary 1.1. *Strongly regular graph with parameters* $(1269, 288, 42, 72)$*, in which neighbourhoods of vertices are strongly-regular graphs with parameters* $(288, 42, 6, 6)$*, is not vertex transitive.*

Theorem 1.3. *Let* Γ *be a distance-regular graph with intersection array* $\{288, 245, 48, 1; 1, 24, 245, 288\}$*,* $G = \mathrm{Aut}(\Gamma)$*,* g *be an elment in* G *with prime order* p *and* $\Omega = \mathrm{Fix}(g)$*. Then* $\pi(G) \subseteq \{2, 3, 5, 7, 47\}$ *and one of the following statements holds:*

(1) Ω *is empty graph and either*
(*i*) $p = 3$*,* $\alpha_4(g) = v$*, or* $\alpha_4(g)$ *divided by 9,* $\alpha_1(g) = 126l + 234 + 2\alpha_4(g)$ *and* $\alpha_3(g) = 252l - 18 - 2\alpha_4(g)$*,* $l \leqslant 3$*, or*
(*ii*) $p = 47$*,* $\alpha_4(g) = 0$*,* $alpha_1(g) = 6 \cdot 47$ *and* $alpha_3(g) = 12 \cdot 47$*;*

(2) Ω *is an union of 3 isolated* n*-cliques and either* $p = 5$*,* $n = 4, 9$*, or* $p = 7$*,* $n = 2, 9$*;*

(3) $p = 3$*,* $\bar{\Omega}$ *is* l*-coclique or it contains geodesic* 2*-path;*

(4) $p = 2$*,* $\bar{\Omega}$ *is* n*-clique,* l*-coclique or it contains geodesic* 2*-path.*

Corollary 1.2. *The group of automorphisms of distance-regular graph with intersection array* $\{288, 245, 48, 1; 1, 24, 245, 288\}$ *acts intransitive on the set of its antipodal classes.*

2. Preliminary results

In this section are some of the preliminary results, used in the proofs of Theorems.

Lemma 2.1 ([2])**.** *Let* Γ *be a strongly regular draph with parameters* (v, k, λ, μ) *and with non-principal eigenvalues* $r, s,\ s < 0$*. If* Δ *is induced regular subgraph from* Γ *of degree* d *on* w *vertices, then*

$$s \leqslant d - \frac{w(k-d)}{v-w} \leqslant r,$$

and one of the inequalities reached if and only if when each vertex from $\Gamma - \Delta$ *is adjacent to exactly* $w(k-d)/(v-w)$ *vertices from* Δ*.*

The proof of Theorems use Higmen's method for investigation automorphisms of distance-regular graphs, represented in third chapter in Cameron's monograph [3]. Let Γ be a distance-regular graph of diameter d with v vertices. Then we have the symmetric association scheme $(X, \mathcal{R})$ with d classes, where X is the set of vertices of Γ and $R_i = \{(u, w) \in X^2 \mid d(u, w) = i\}$. For vertex $u \in X$ set $k_i = |\Gamma_i(u)|$. Let A_i be the adjacency matrix of the graph Γ_i. Then $A_iA_j = \sum p^l_{ij}A_l$ for some integer numbers $p^l_{ij} \geqslant 0$, which are called the intersection numbers. Note that $p^l_{ij} = |\Gamma_i(u) \cap \Gamma_j(w)|$ for every vertices u, w with $d(u, w) = l$.

Let P_i be the matrix in which in the (j, l) entry there is p^l_{ij}. Then the eigenvalues $k = p_1(0), \ldots, p_1(d)$ of the matrix P_1 are eigenvalues of Γ with multiplicities $m_0 = 1, \ldots, m_d$. Note that the matrix P_i is the value of some integer polinom of P_1, so the ordering of eigenvalues of the matrix P_1 gives the ordering of eigenvalues of P_i. The matrices P and Q with (i, j) entry $p_j(i)$ and $Q_{ji} = m_jp_i(j)/k_i$ are called the first and the second eigenmatrix of Γ and $PQ = QP = vI$, where I is the identity matrix of order $d+1$. Let u_j and w_j be the left and the right eigenvectors of matrix P_1 affording eigenvalue $p_1(j)$ and having the first coordinate 1. Then the multiplicity m_j of the eigenvalue $p_1(j)$ is equal $v/\langle u_j, w_j\rangle$. In fact, w_j are the columns of the matrix P and m_ju_j are the rows of the matrix Q.

The permutation representation of the group $G = \mathrm{Aut}(\Gamma)$ on the vertex set of Γ naturally gives the matrix representation ψ of G in $GL(v, \mathbb{C})$. The space $\mathbb{C}^v$ is the orthogonal direct sum of the eigenspaces $W_0, W_1, ..., W_d$ of the adjacent matrix $A = A_1$ of Γ. For every $g \in G$ we have $\psi(g)A = A\psi(g)$, so the subspace W_i is $\psi(G)$-invariant. Let χ_i be a character of the representation ψ_{W_i}. Then for $g \in G$ we obtain $\chi_i(g) = v^{-1} \sum_{j=0}^{d} Q_{ij}\alpha_j(g)$, where $\alpha_j(g)$ is the numbers of vertices x of X such that $d(x, x^g) = j$.

Lemma 2.2 ([4], Lemmas 1–2). *Let Γ be a distance-regular graph with i-th integer eigenvalue θ_i, ψ be a monomial representation of a group $G = \mathrm{Aut}(\Gamma)$ to the group of linear transformations of the space $V = \mathbf{C}^v$, χ_i be a character of projection ψ to subspace W_i of dimensionality m_i, generated by the eigenvectors of the adjacency matrix of the graph Gamma, corresponding to theta$_i$. Then the following statements hold:*

(1) *if Q is rational matrix, then for element g from G we have $\alpha_i(g) = \alpha_i(g^l)$ for every l, relatively prime with $|g|$;*

(2) *if g is element from G with prime order p, then p divides $m_i - \chi_i(g)$;*

(3) *if g is element from G with order p, p is prime number, then p^2 divides $m_i - \chi_i(g^p)$.*

Lemma 2.3 ([5], Theorem 3.2). *Let Γ be a strongly-regular graph with parameters (v, k, λ, μ) and with eigenvalues $k, r, -m$. If g is automorphism of Γ and $\Omega = \mathrm{Fix}(g)$, then $|\Omega| \leqslant v \cdot \max\{\lambda, \mu\}/(k - r)$.*

3. Automorphisms of a graph with parameters $(288, 42, 6, 6)$

In this section we assume, that Γ is strongly-regular graph with parameters $(288, 42, 6, 6)$ and with spectrum $42^1, 6^{140}, -6^{147}$, $G = \mathrm{Aut}(\Gamma)$, g is element of G with prime order p $\Omega = \mathrm{Fix}(g)$. By Lemma 2.3 we have $|\Omega| \leqslant 288 \cdot 6/36 = 48$.

Lemma 3.1. *Let χ_1 be a character of representation ψ on subspace of dimension* 140. *Then*

(1) $\chi_1(g) = (6\alpha_0(g) + \alpha_1(g))/12 - 4$ *and* $\chi_1(g) - 140$ *is divided by p;*

(2) *if Ω is empty graph, then either $p = 2$, $\alpha_1(g) = 24s$, or $p = 3$, $\alpha_1(g) = 36t$;*

(3) *if Ω is n-clique, then either $p = 7$, $n = 1$, $\alpha_1(g) = 84s + 42$ or $n = 8$, $\alpha_1(g) = 84s$, or $p = 5$, $n = 3$, $\alpha_1(g) = 60s + 30$ or $n = 8$, $\alpha_1(g) = 60s$;*

(4) *if Ω is l-coclique,then either $p = 2$, l is even, $6 \leqslant l \leqslant 36$ and $\alpha_1(g) = 24t - 6l$, or $p = 3$, l is divided by 3, $3 \leqslant l \leqslant 36$ u $\alpha_1(g) = 36t - 6l$.*

Proof. We have
$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 140 & 20 & -4 \\ 147 & -21 & 3 \end{pmatrix}.$$

Therefore $\chi_1(g) = 1/72(35\alpha_0(g) + 5\alpha_1(g) - \alpha_2(g))$. As $\alpha_2(g) = v - \alpha_0(g) - \alpha_1(g)$, then $\chi_1(g) = (6\alpha_0(g) + \alpha_1(g))/12 - 4$. Finally, $\chi_1(g) - 140$ is divided by p by Lemma 2.2.

Let Ω be an ampty graph. Then $p \in \{2, 3\}$. If $p = 2$, then $\chi_1(g)$ is even, so $\alpha_1(g) = 24s$. If $p = 3$, then $\alpha_1(g) = 36t$.

Let Ω be a n-clique. By the Hoffman boundary we have maximal order of clique in Γ is not any more than $n \leqslant 1 + k/m = 8$. If $n = 1$, then p divides 140 and 147, so $p = 7$, $\chi_1(g) = (6 + \alpha_1(g))/12 - 4$ and $\alpha_1(g) = 84s + 42$.

If $n \geqslant 2$, then p divides $8 - n$, 35 and 210, so either $p = 5$, $n = 3$, $\chi_1(g) = (18 + \alpha_1(g))/12 - 4$ and $\alpha_1(g) = 60s + 30$ or $n = 8$, $\chi_1(g) = \alpha_1(g)/12$ and $\alpha_1(g) = 60s$, or $p = 7$, $n = 8$, $\chi_1(g) = \alpha_1(g)/12$ and $\alpha_1(g) = 84s$.

Let Ω be a l-coclique, $l \geqslant 2$. By the Hoffman boundary for cocliques, we have $l \leqslant vm/(k+m) = 36$. Further, p divides 6 and $210 - l$, so $p = 2, 3$. If $p = 2$, then l is even, any vertex in $\Gamma - \Omega$ is adjacent to even number of vertices in Ω, $l \geqslant 6$, $\chi_1(g) = (6l + \alpha_1(g))/12 - 4$ and $\alpha_1(g) = 24t - 6l$. If $p = 3$, then l divided by 3 and $\alpha_1(g) = 36t - 6l$. In case $l = 36$ any vertex in $\Gamma - \Omega$ is adjacent to exactly 6 vertices in Ω, so $\alpha_1(g) = 0$.

Let Ω contains an edge and Ω be an union of isolated cliques. Then p divides 35 and 36, contradiction. □

Lemma 3.2. *Let Ω contains geodesic 2-path b, a, c. Then*

(1) *Γ does not contain its own strongly regular subgraph Δ with parameters $(v', k', 6, 6)$;*

(2) *Ω does not contain vertices of degree 42;*

(3) *$p \leqslant 5$ and if $p = 5$, then $|\Omega| = 5t + 3$, $t = 2, 3, \ldots, 9$;*

(4) *if $p = 3$, then $|\Omega| = 3s$, $s = 2, 3, \ldots, 16$, and in the case $s = 2$ Ω is complete bipartite graph $K_{3,3}$;*

(5) *if $p = 2$, then $|\Omega| = 2l$, $l = 2, 3, \ldots, 24$, and in the case $l = 2$ Ω is quadrangle.*

Proof. Assume that Γ contains its own strongly regular subgraph Δ with parameters $(v', k', 6, 6)$. Then $n^2 = 4(k' - 6)$, $n = 2u$, $k' = u^2 + 6$ and Δ has nonprincipal eigenvalues $u, -u$. Note that each vertex in $\Gamma - \Delta$ is adjacent to at most one vertex of Δ. Further, the multiplicity of u is $f = (u-1)(u^2+6)(u^2+u+6)/(12u)$, so either $u = 2$, $f = 5$, $k' = 10$ and $v' = 16$, or $u = 3$, $f = 15$, $k' = 15$ and $v' = 36$. In any case, a vertex of $\Gamma - \Delta$ is adjacent to more than one vertex of Δ, contradiction.

If $p \geqslant 7$, then Ω is strongly-regular graph with parameters $(v', k', 6, 6)$, contradiction.

If Ω contains vertex a of degree 42, than for each vertex $u \in \Gamma - a^\perp$ subgraph $[u] \cap \Omega$ contains 6 vertices in $[a]$. For $u \notin \Omega$ we have $[u] \cap \Omega = [u] \cap [a]$. If $b \in \Omega - a^\perp$, than $[b]$ contained in Ω. This contradicts the fact that each vertex of $\Gamma - (a^\perp \cup \Omega)$ is adjacent to 6 vertices in $[u] \cap [a]$. So, $|\Omega| = 43$ and $\alpha_1(g) = 0$ (otherwise the vertex u, adjacent to u^g, the subgraph $[u] \cap [u^{g^2}]$ contains u^g and 6 vertices of Ω, contradiction). But then $\chi_1(g) = 258/12 - 4$, contradiction.

Let $p = 5$. Then $\lambda_\Omega, \mu_\Omega \in \{1, 6\}$, $|\Omega| = 5t + 3$, $t \leqslant 9$ and degrees of vertices in Ω are $2, 7, \ldots, 37$. If $t = 1$, then $\Omega = a^\perp$ for vertex a of degree 7 of Ω and Ω contains even number of vertices of degree 7. If Ω contains two vertices a, b of degree 7, then $\Omega - \{a, b\}$ is coclique. This contradicts the fact that $\mu_\Omega \in \{1, 6\}$. So, Ω is regular graph of valency 2 with $\lambda_\Omega = \mu_\Omega = 1$, contradiction.

Let $p = 3$. Then $\lambda_\Omega, \mu_\Omega \in \{0, 3, 6\}$, $|\Omega| = 3s$, $s \leqslant 16$ and degrees of vertices in Ω are $0, 3, \ldots, 39$. If $s = 2$, then Ω is complete bipartite graph $K_{3,3}$.

Let $p = 2$. Then $\lambda_\Omega, \mu_\Omega \in \{0, 2, 4, 6\}$, $|\Omega| = 2l$, $l \leqslant 24$ and degrees of vertices in Ω are $0, 2, \ldots, 40$. If $|\Omega| = 4$, then Ω is quadrangle. □

Lemmas 3.1, 3.2 imply the Theorem 1.1.

4. Automorphisms of graph with parameters $(1269, 288, 42, 72)$

In this section we assume, that Γ is strongly-regular graph with parameters $(1269, 288, 42, 72)$ and with spectrum $288^1, 6^{1080}, -36^{188}$, in which neighborhoods of vertices are strongly regular with parameters $(288, 42, 6, 6)$, $G = \mathrm{Aut}(\Gamma)$, g is elment of G with prime order p and $\Omega = \mathrm{Fix}(g)$. By Lemma 2.3 we have $|\Omega| \leqslant 1269 \cdot 72/282 = 324$.

Lemma 4.1. *Let χ_2 be a character of representation ψ on subspace of dimension* 188. *Then*

(1) *$\chi_2(g) = (6\alpha_0(g) - \alpha_1(g))/42 + 47/7$ and $\chi_2(g) - 188$ divided by p;*

(2) *if Ω is empty graph, then either $p = 47$ and $\alpha_1(g) = 47\cdot 6$, or $p = 3$ and $\alpha_1(g) = 126l+72$;*

(3) *if $[a] \subset \Omega$, then $\alpha_1(g) = 0$, $a^\perp = \Omega$ and $p = 2$;*

(4) *if Ω is n-clique, then either $p = 2$, $n = 1$, $\alpha_1(g) = 84t + 36$, or $p = 5$, $n = 4$, $\alpha_1(g) = 210s+180$ or $n = 9$, $\alpha_1(g) = 210s$, or $p = 7$, $n = 2$, $\alpha_1(g) = 294t+42$ or $n = 9$, $\alpha_1(g) = 294t+84$;*

(5) *if Ω is l-coclique, then either $p = 2$, l is odd, $\alpha_1(g) = 84m + 6l + 282$, or $p = 3$, l divided by 3, $\alpha_1(g) = 126m + 6l + 324$.*

Proof. For $i > 1$ we denote $\alpha_i(g) = pw_i$. Than we have

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1080 & 45/2 & -54/7 \\ 188 & -47/2 & 47/7 \end{pmatrix}.$$

So $\chi_2(g) = 1/27(4\alpha_0(g) - \alpha_1(g)/2 + \alpha_2(g)/7)$. As $\alpha_2(g) = v - \alpha_0(g) - \alpha_1(g)$, then $\chi_2(g) = (6\alpha_0(g) - \alpha_1(g))/42 + 47/7$. Finally, $\chi_2(g) - 188$ divided by p by Lemma 2.2.

Let Ω be an empty graph. Then $p \in \{3, 47\}$. If $p = 47$, then $\chi_2(g) = -\alpha_1(g)/42 + 47/7$, so $\alpha_1(g) = 47 \cdot 6$. If $p = 3$, then $\chi_2(g) = (-w_1 + 94)/14$ and $\alpha_1(g) = 126l + 72$.

Now, in view of Theorem 1.1, we have $\pi(G) \subseteq \{2, 3, 5, 7, 47\}$.

If $[a] \subset \Omega$, then for each vertex $u \in \Gamma - \Omega$ subgraph $[u] \cap \Omega$ contains 72 vertices of $[a]$, so each $\langle g\rangle$-orbit on $\Gamma - \Omega$ is coclique and $\alpha_1(g) = 0$. Further, $a^\perp = \Omega$, $\chi_2(g) - 188 = -140$ divided by p, so $p = 2, 5, 7$. Note than any vertex $u \in \Gamma - \Omega$ is sdjacent to at most one vertex of each $\langle g\rangle$-orbit of lenght p. There are 140 orbits of length 7 and 196 orbits of length 5, which contradicts the fact that $72 + 139$ and $72 + 195$ is less than 288. So, $p = 2$.

Let Ω be a n-clique. By the Hoffman boundary we have maximal order of clique is not any more than $n \leqslant 1 + k/m = 9$. If $n = 1$, then p divides 288 and 980, so $p = 2$, number $\chi_2(g) = (144 - w_1)/21$ is even and so $\alpha_1(g) = 84t + 36$. If $n \geqslant 2$, then p divides $44 - n$ and 245, so either $p = 5$, $n = 4$, $\alpha_1(g) = 210s + 180$ or $n = 9$, $\alpha_1(g) = 210s$, or $p = 7$, $n = 2$, $\chi_2(g) = 9 - 7w_1/42$, $\alpha_1(g) = 294t + 42$ or $n = 9$, $\chi_2(g) = 8 - \alpha_1(g)/42$ and $\alpha_1(g) = 294t + 84$.

Let Ω be a l-coclique, $l \geqslant 2$. By the Hoffman boundary we have maximal order of coclique is not any more than $l \leqslant vm/(k + m) = 141$. Further, p divides 72 and $765 - l$, so either $p = 2$, l is odd, number $\chi_2(g) = (l + 47 - \alpha_1(g)/6)/7$ is even and $\alpha_1(g) = 84m + 6l + 282$, or $p = 3$, l divided by 3, number $\chi_2(g) = (l + 47 - \alpha_1(g)/6)/7$ comparable to -1 by modulo 3 and $\alpha_1(g) = 126m + 6l + 324$.

Let Ω contains an edge and Ω be an union of isolated cliques. Then p divides 245 and 72, contradiction. □

Lemma 4.2. *Let Ω contains geodesic 2-path b, a, c. Then*

(1) *Γ does not contain its own strongly-regularsubgraphs with parameters $(v', k', 42, 72)$;*

(2) *$p = 5$, $|\Omega| = 5l + 4$, $l \leqslant 64$, for $e \in \Omega$ we have $|\Omega(e)| = 5t + 3$, $t = 2, 3, \dots, 9$, $210m + 30l + 180$;*

(3) *$p = 3$, $|\Omega| = 3n$, $n \leqslant 108$, for $e \in \Omega$ we have $|\Omega(e)| = 3t$, $t = 1, 2, \dots, 8$, $\alpha_1(g) = 18n + 126l + 324$;*

(4) *$p = 2$, $|\Omega| = 2l+1$, $l \leqslant 161$, for $e \in \Omega$ we have $|\Omega(e)| = 2n$, $n \leqslant 12$, $\alpha_1(g) = 12l+84s+288$.*

Proof. Assume Γ contains its own strongly-regular subgraph Δ with parameters $(v', k', 42, 72)$. Then $n^2 = 900 + 4(k' - 72)$, $n = 2u$, $k' = u^2 - 153$ and Δ has nonprincipal eigenvalues $u - 15, -(u + 15)$. Further, multiplicity of $u - 15$ is $f = (u + 14)(u^2 - 153)(u^2 + u - 138)/(144u)$, contradiction.

By Theorem 1.1 we have $p = 2, 3, 5$.

Let $p = 5$. Then $|\Omega| = 5l + 4$, $l \leqslant 64$ and by theorem 1 subgraph $\Omega(a)$ contains geodesic 2-path, $|\Omega(a)| = 5t + 3$, $t = 2, 3, \ldots, 9$. Finally, the numder $\chi_2(g) = (5l + 51 - \alpha_1(g)/6)/7$ is comparable to 3 by modulo 5 and $\alpha_1(g) = 210m + 30l + 180$.

Let $p = 3$. Then $|\Omega| = 3n$, $n \leqslant 108$ and by theorem 1 we have $|\Omega(a)| = 3t$, $t = 1, 2, \ldots, 8$, the number $\chi_2(g) = (3n + 47 - \alpha_1(g)/6)/7$ is comparable to -1 by modulo 3 and $\alpha_1(g) = 18n + 126l + 324$.

Let $p = 2$. Then $|\Omega| = 2l + 1$, $l \leqslant 161$ and by theorem 1 we have $|\Omega(a)| = 2n$, $n \leqslant 12$, the number $\chi_2(g) = (2l + 48 - \alpha_1(g)/6)/7$ is even and $\alpha_1(g) = 12l + 84s + 288$. □

Lemmas 4.1, 4.2 imply the Theorem 1.2.

Lemma 4.3. *Let the group* $G = \mathrm{Aut}(\Gamma)$ *acts transitively on the set vertices of* Γ. *Then the following assertions hold:*

(1) *if* f *is element of* G *with order* 47, *then* $|C_G(f)|$ *divides* 94;

(2) $S(G) = 1$;

(3) *if* T *is socle of group* G, *then* $|T|$ *divided by* 23.

Proof. Let f be an element of G with order 47, g be an element of $C_G(f)$ with order $p < 47$. By theorem 2 $\mathrm{Fix}(f)$ is empty graph and $\alpha_1(f) = 282$. In view of Theorem 2 and the action of f on Ω should be one of the statements:

(1) Ω is empty graph, $p = 3$ and $\alpha_1(g) = 126l + 72$;

(2) Ω is l-coclique, $p = 2$, l is odd, $\alpha_1(g) = 84m + 6l + 282$ or $p = 3$, l divided by 3, $\alpha_1(g) = 126m + 6l + 324$;

(3) Ω contains geodesic 2-path and either
(*i*) $p = 5$, $|\Omega| = 5l + 4$, $l \leqslant 64$, $\alpha_1(g) = 210m + 30l + 180$, or
(*ii*) $p = 3$, $|\Omega| = 3n$, $n \leqslant 108$, $\alpha_1(g) = 18n + 126l + 324$, or
(*iii*) $p = 2$, $|\Omega| = 2l + 1$, $l \leqslant 161$, $\alpha_1(g) = 12l + 84s + 288$.

If Ω is empty graph, then $\alpha_1(g) = 18(7l + 4)$ divided by 47, contradiction. If Ω is nonempty graph, then $|\Omega| = 47t$, p divides $27 - t$ and either $t = 6$, $p = 3, 7$, or $t = 5, 1$, $p = 2$, or $t = 3$, $p = 2, 3$, or $t = 2$, $p = 5$. Further, $\chi_2(g) = (282t - \alpha_1(g))/42 + 47/7$, $\alpha_1(g) = 282w$ and $\chi_2(g) = 47(t - w + 1)/7$. In case $p = 5$ we have $\alpha_1(g) = 47 \cdot 30l = 0$, $t = 2$ and $\chi_2(g) = 47 \cdot 3/7$, contradiction. If Ω is coclique, then by Hoffman's boundary we have $|\Omega| \leqslant 141$, so $|\Omega| = 141$ and $4 + w$ divided by 7. Now, each vertex of $\Gamma - \Omega$ is adjacent to 36 vertices of Ω. This contradicts the fact that for the 3-clique $\{u, u^g, u^{g^2}\}$ neighborhood of vertex u in Γ is strongly-regular graph with parameters $(288, 42, 6, 6)$.

Let Ω contains geodesic 2-path. If $p = 3$, then $|\Omega| = 3n$, $n = 47, 94$ and $\alpha_1(g) = 18(n+7l+18)$ divided by 47, contradiction. If $p = 2$, then $|\Omega| = 2l+1$, $l = 23, 60, 117$ and $\alpha_1(g) = 12(l+7s+24)$ divided by 47, so $l = 23$ and $s = 0$.

Note, that $|\Gamma - \Omega| = 26 \cdot 47$ is not divisible by 4, so $|C_G(f)|$ is not divisible by 4.

As $v = 27 \cdot 47$, then by assertion (1) either $S(G) = O_3(G)$, or $G/O_3(G)$ contains normal subgroup with order 47. Let $Q = O_3(G) \neq 1$, f be an element of G with order 47. Then $|Q : Q_a| = 27$, and as $k = 288$, then $|Q_a : Q_{a,b}| = 9$ for any vertex $b \in [a]$. Further, the order of Sylow 3-subgroup of symmetric group of degree 42 is 3^{19}, and by the theorem 1 the number $|Q_{a,b}|$ divides 3^{19}. As $3^{23} - 1$ divided by 47, and $3^i - 1$ is not divisible by 47 for $i < 23$, then Q is elementary Abelian group of order 3^{23}. On the other hand, the group $Q_{a,b}$ acts faithfully on $[a] \cap [b]$ and for $c \in [a] \cap [b]$ group $Q_{a,b,c}$ has an index that divides 3 in $Q_{a,b}$, and acts on $([a] \cap [b] \cap [c]) \cup ([a] \cap [b] - [c])$. Hence the subgroup of index 81 from $Q_{a,b,c}$ is embedded in a Sylow 3-subgroup of the symmetric group of degree 27. This contradicts the fact that $|Q_{a,b,c}|$ divided by 3^{18}, and order of Sylow 3-subgroup of the symmetric group of degree 27 is 3^{13}.

So, $O_3(G) = 1$. By the table 1 from [6] the order of socle T of group G is divided by 23. $\square$

As in wiew of Theorem 2 $|G|$ is not divisible by 23, then G acts intransitive on the set of vertices of graph Γ. Corollary 1.1 is proved.

5. Automorphisms of graph with intersection array $\{288, 245, 48, 1; 1, 24, 245, 288\}$

Let to the end of paper Γ be a distance-regular graph with intersection array $\{288, 245, 48, 1; 1, 24, 245, 288\}$ and with spectrum $288^1, 48^{282}, 6^{1080}, -6^{2256}, -36^{188}$, $G = \mathrm{Aut}(\Gamma)$, g be an element of G with prime order p and $\Omega = \mathrm{Fix}(g)$.

Lemma 5.1. *Let χ_1 be a character of representation ψ on subspace of dimension 282, χ_4 be a character of representation ψ on subspace of dimension 188. Then*

(1) *$\chi_1(g) = (12\alpha_0(g) + 2\alpha_1(g) - \alpha_3(g) - 6\alpha_4(g))/162$, $\chi_1(g) - 282$ divided by p;*

(2) *$\chi_4(g) = (7\alpha_0(g) + alpha_2(g) + 7\alpha_4(g))/126 - 47/2$, $\chi_4(g) - 188$ divided by p;*

(3) *if Ω is empty graph, then either*
(i) $p = 3$, $\alpha_4(g) = v$, or
(ii) $p = 3$, $\alpha_4(g)$ divided by 9, $\alpha_1(g) = 126l + 234 + 2\alpha_4(g)$ and $\alpha_3(g) = 252l - 18 - 2\alpha_4(g)$, $l \leqslant 3$, or
(iii) $p = 47$, $\alpha_4(g) = 0$, $alpha_1(g) = 6 \cdot 47$ and $alpha_3(g) = 12 \cdot 47$.

Proof. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 282 & 47 & 0 & -47/2 & -141 \\ 1080 & 45/2 & -54/7 & 45/2 & 1080 \\ 2256 & -47 & 0 & 47/2 & -1128 \\ 188 & -47/2 & 47/7 & -47/2 & 188 \end{pmatrix}.$$

So $\chi_1(g) = (12\alpha_0(g) + 2\alpha_1(g) - \alpha_3(g) - 6\alpha_4(g))/162$, $\chi_1(g) - 282$ divided by p.

Similarly, $\chi_4(g) = (56\alpha_0(g) - 7\alpha_1(g) + 2\alpha_2(g) - 7\alpha_3(g) + 56\alpha_4(g))/189$. As $\alpha_1(g) + \alpha_3(g) = v - \alpha_0(g) - \alpha_2(g) - \alpha_4(g)$, then $\chi_4(g) = (7\alpha_0(g) + alpha_2(g) + 7\alpha_4(g))/126 - 47/2$, $\chi_4(g) - 188$ divided by p.

Let Ω is empty graph. Then $p \in \{3, 47\}$. If $p = 3$, then either $\alpha_4(g) = v$, or g induces an automorphism of order 3 of antipodal private $\bar{\Gamma}$. In last case we have $\chi_4(g) = (alpha_2(g) + 7\alpha_4(g))/126 - 47/2$. As $\alpha_4(g) = 3w_4$, then $\alpha_2(g) = 21w_2$ and $\chi_4(g) = (w_2 + w_4)/6 - 47/2$ comparable to -1 by modulo 3. Further, $\chi_1(g) = (2\alpha_1(g) - \alpha_3(g) - 6\alpha_4(g))/162$ divided by. In other hand, $\bar{\alpha}_0(g) = \alpha_4(g)/3$ divided by 3 and by theorem 2 we have $\bar{\alpha}_1(g) = (\alpha_1(g) + \alpha_3(g))/3 = 126l + 72$. So, $\alpha_1(g) = 126l + 234 + 2\alpha_4(g)$ and $\alpha_3(g) = 252l - 18 - 2\alpha_4(g)$, $l \leqslant 3$.

If $p = 47$, then $\alpha_4(g) = 0$, $\chi_4(g) = alpha_2(g)/126 - 47/2$, $\chi_1(g) = (2\alpha_1(g) - \alpha_3(g))/162$ are divided by 3. So, $alpha_2(g) = 63 \cdot 47$ and $alpha_1(g) + alpha_3(g) = 18 \cdot 47$, and so $alpha_1(g) = 6 \cdot 47$, $alpha_3(g) = 12 \cdot 47$.

$\chi_4(g) = (\alpha_2(g) + 20l - 460)/40$, the number $\chi_4(g)$ is odd and $\alpha_2(g) = 20 - 20l - 80s$. Further, $\alpha_1(g) + \alpha_3(g) = 624 + 20l + 80s$, the number $\chi_1(g) = (\alpha_1(g) - 312 - 18l - 40s)/28$ is even, so $\alpha_1(g) = 56m + 18l + 40s + 32$, $\alpha_3(g) = 592 + 2l + 40s - 56m$.

If $p = 7$, then $\alpha_4(g) = 0$, $\chi_4(g) = \alpha_2(g)/40 - 23/2$, $\alpha_2(g) = 280l + 140$. Further, $\alpha_1(g) + \alpha_3(g) = 504 - 280l$, $\chi_1(g) = (\alpha_1(g) - 252 + 140l)/28$, the number $\chi_1(g)$ comparable to 4 by modulo 7, so $\alpha_1(g) = 140 - 140l + 196t$ and $\alpha_3(g) = 364 - 196t - 140l$.

If $p = 23$, then $\alpha_4(g) = 0$, $\chi_4(g) = \alpha_2(g)/40 - 23/2$, $\alpha_2(g) = 460$. Further, $\alpha_1(g) + \alpha_3(g) = 184$, $\chi_1(g) = (\alpha_1(g) - 92)/28$, so $\alpha_1(g) = \alpha_3(g) = 92$. $\square$

Lemma 5.2. *If element g induces a nontrivial automorphism of the graph $\bar\Gamma$ and Ω is non-empty graph, then one of the following assertions hold:*

(1) *Ω is an union of three isolated n-cliques and either $p = 5$, $n = 4, 9$, or $p = 7$, $n = 2, 9$;*

(2) *$p = 2$, $\bar\Omega$ is a n-clique, l-coclique or contains geodesic 2-path;*

(3) *$p = 3$, $\bar\Omega$ is a l-coclique or contains geodesic 2-path.*

Proof. Let g induces a nontrivial automorphism of $\bar\Gamma$. By Theorem 2 either $\bar\Omega$ is empty graph, $p = 3, 47$, or $\bar\Omega$ is clique or coclique, or $\bar\Omega$ contains geodesic 2-path.

If $\bar\Omega$ is empty graph, then Ω is emty graph too.

If $\bar\Omega$ is n-clique, then by Theorem 2 either $p = 2$, $n = 1$, $\bar\alpha_1(g) = 84t + 36$, or $p = 5$, $n = 4$, $\bar\alpha_1(g) = 210s + 180$ or $n = 9$, $\bar\alpha_1(g) = 210s$, or $p = 7$, $n = 2$, $\bar\alpha_1(g) = 294t + 42$ or $n = 9$, $\bar\alpha_1(g) = 294t + 84$.

In case $p = 2$ we have $\alpha_0(g) + \alpha_4(g) = 3$, the number $\chi_4(g) = (21 + alpha_2(g))/126 - 47/2$ is even, $\alpha_2(g) = 126s + 42$ and by Theorem 2 we have $\bar\alpha_2(g) = 1232 - 84t = 42s + 14$. Further, $\alpha_1(g) + \alpha_3(g) = 3762 - 126s$, the number $\chi_1(g) = (6\alpha_0(g) + \alpha_1(g) - 1274)/54$ is even, $\alpha_1(g) = 108l + 1274 - 6\alpha_0(g)$ and $\alpha_3(g) = 2498 - 108l + 6\alpha_0(g) - 126s$.

In case $p = 5$ we have $\alpha_4(g) = 0$, $\alpha_0(g) = 3n$, the number $\chi_4(g) = (21n+alpha_2(g))/126-47/2$ comparable to 3 by modulo 5, $\alpha_2(g) = 126t + 63 - 21n$. If $n = 4$, then by Theorem 2 we have $\bar\alpha_2(g) = 1085 - 210s$ and $\chi_4(g) = (159 - 30s)/6 - 47/2 = 3 - 5s$. If $n = 9$, then by Theorem 2 we have $\bar\alpha_2(g) = 1260 - 210s$ and $\chi_4(g) = (189 - 30s)/6 - 47/2 = 8 - 5s$.

In case $p = 7$ we have $\alpha_4(g) = 0$, $\alpha_0(g) = 3n$, the number $\chi_4(g) = (21n+alpha_2(g))/126-47/2$ comparable to 6 by modulo 7. If $n = 2$, then by Theorem 2 we have $\bar\alpha_2(g) = 1225 - 294t$ and $\chi_4(g) = (59 - 14t - 47)/2$. If $n = 9$, then by Theorem 2 we have $\bar\alpha_2(g) = 1176 - 294t$ and $\chi_4(g) = (59 - 14t)/2 - 47/2$.

Let $\bar\Omega$ be a l-coclique and $p = 2$. Then l is odd, $\bar\alpha_2(g) = 987-84m-7l$, $\chi_4(g) = (141-12m)/6-47/2 = -2m$. Let $\bar\Omega$ be a l-coclique and $p = 3$, l is divided by 3, $\bar\alpha_2(g) = 1269 - 126m - 7l - 324$, $\chi_4(g) = (45 - 6m - 47)/2$.

Let $\bar\Omega$ contains geodesic 2-path. If $p = 5$, then $|\bar\Omega| = 5l+4$, $l \leqslant 64$, $\bar\alpha_2(g) = 1085-210m-35l$, $\chi_4(g) = (33 - 10m - 47)/2$. If $p = 3$, then $|\bar\Omega| = 3n$, $n \leqslant 108$, $\bar\alpha_2(g) = 945 - 21n - 126l$, $\chi_4(g) = (45 - 6l - 47)/2$. If $p = 2$, then $|\bar\Omega| = 2l + 1$, $l \leqslant 161$, $\bar\alpha_2(g) = 980 - 14l - 84s$, $\chi_4(g) = -2s$. □

Lemmas 5.1, 5.2 imply the Theorem 1.3. Theorem 1.3 and Corollary 1.1 imply the Corollary 1.2.

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Автоморфизмы AT4$(6,6,3)$-графа и отвечающих ему сильно регулярных графов

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Кулен и Юришич определили класс AT4-графов (антиподальных плотных графов диаметра 4). Среди этих графов имеется граф с массивом пересечений $\{288,245,48,1;1,24,245,288\}$ *на* $v = 1+288+2940+576+2 = 3807$ *вершинах. Антиподальное частное этого графа является сильно регулярным графом с параметрами* $(1269,288,42,72)$*. Оба этих графа являются локально псевдо* $GQ(7,5)$*-графами. В работе найдены возможные автоморфизмы указанных графов. В частности, группа автоморфизмов дистанционно регулярного графа с массивом пересечений* $\{288,245,48,1;1,24,245,288\}$ *действует интранзитивно на множестве его антиподальных классов.*

Ключевые слова: дистанционно регулярный граф, сильно регулярный граф, автоморфизм графа.